

Thermodynamical limit in non-extensive and Rényi statistics

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Previous results on Rényi and Wang's formalism of the Tsallis thermostatics are founded by using an extensive variable z connected to the entropic parameter q . It is shown that in the thermodynamical limit both thermostatics meet all the requirements of equilibrium thermodynamics. In particular, both the Tsallis and Rényi entropies are extensive functions of state and the temperature of the system is intensive. In the thermodynamical limit Wang's incomplete nonextensive statistics resembles the Tsallis one, but the Rényi thermostatics is reduced to the usual Boltzmann-Gibbs one. The principle of additivity and the zeroth law of thermodynamics in the canonical ensemble for both thermostatics are demonstrated on the particular example of the classical ideal gas of identical particles.

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I. INTRODUCTION

The statistical mechanics based on the nonextensive Tsallis entropy [1, 2] finds support in recent studies due to non-Gibbs phase distribution functions which appear to be successful in analyzing certain experimental data [3, 4]. In these investigations the statistical Boltzmann-Gibbs entropy is modified by the additional parameter q , appearing both in the distribution function and in the equation of state of the system. They enlarge the range of application of the usual statistical mechanics without taking into account realistic interactions in the microscopic Hamiltonian. In this respect two problems occur: the problem of the physical interpretation of the parameter q and the problem of connecting such statistical treatments to equilibrium thermodynamics [5]. The latter gives rise to theoretical discussions in the literature due to the difficulties in the proof of the zeroth law of thermodynamics. It is closely connected to the principle of additivity. In the Tsallis statistics the principle of additivity is violated as the statistical entropy is nonextensive due to its definition. Therefore, in the case when the parameter q is a universal constant all attempts to proof the zeroth law of thermodynamics for finite systems failed [6, 7, 8, 9, 10]. Moreover, in this case the thermodynamical limit is incompatible with the Gibbs limit, $q \rightarrow 1$ [11]. It is well known that equilibrium thermodynamics is a macroscopic, phenomenological theory defined only in the thermodynamical limit [12]. Therefore, an unambiguous connection between statistical mechanics and equilibrium thermodynamics can be provided only in the thermodynamical limit, while boundary effects can be neglected. Thus, first of all, the correct definition of the thermodynamical limit for the Tsallis statistics and the interpretation of the parameter q must be clarified.

A correct definition of the thermodynamical limit for the Tsallis thermostatics was given first in Botet et al. [13, 14]. Later, in [15], it was revealed that if the parameter $1/(q - 1)$ can be interpreted as an extensive variable of state, then in the microcanonical ensemble the Tsallis entropy becomes extensive in the thermodynamical limit. Thus, the zeroth law of thermodynamics and the principle of additivity are restored, all functions of state are either extensive or intensive in conformity with the requirements of equilibrium thermodynamics. For the canonical ensemble similar results were obtained in the framework of the classical ideal gas of identical particles only [16], because in this case the functions of state can be analytically integrated, and so exact results can be obtained. Note that the procedure stated in [15, 16] assumes the regularization of the final integrated expressions of the functions of state, while they depend only on the variables of state of the system by applying the thermodynamical limit condition. It allows to establish an unambiguous connection of statistical mechanics to equilibrium thermodynamics and to explore the validity of this statistical mechanics using a given entropy.

In our previous paper [17] the microcanonical and canonical ensembles of the Wang's formalism for the Tsallis and the corresponding ensembles for the Rényi entropy were compared. It was shown that in Wang's formalism for the Tsallis statistics both in the microcanonical and the canonical ensembles the zero-th law of thermodynamics and the principle of additivity are violated, as long as the parameter q were a universal constant. This conclusion is also valid for the Rényi statistics in the canonical ensemble. The microcanonical ensemble of the Rényi statistics on the other hand coincides with the Gibbs one. Hence, all laws of the equilibrium thermodynamics are satisfied. In the present paper we show that if the entropic parameter q is connected with an extensive variable of state z in a certain way, then in the thermodynamic limit [15, 16] both the Rényi and Wang's incomplete nonextensive statistics become extensive and they satisfy all laws of equilibrium thermodynamics.

This paper is organized as follows. In the second and third sections our results for the microcanonical and canonical ensembles for Wang's incomplete nonextensive statistics and Rényi one are given. In the thermodynamical limit the connection from statistical mechanics to thermodynamics is implemented.

II. INCOMPLETE NONEXTENSIVE STATISTICS

The incomplete nonextensive thermostatics or the Wang's formalism of the generalized statistical mechanics is based on Tsallis' definition of the statistical entropy [1], and on the incomplete normalization condition for the phase distribution function [7, 18],

$$S = -k \int d\Gamma \frac{\varrho - \varrho^q}{1 - q}, \quad \int d\Gamma \varrho^q = 1. \quad (1)$$

This is conform with a modified expectation value of a dynamical variable A

$$\langle A \rangle = \int d\Gamma A \varrho^q. \quad (2)$$

Here ϱ^q is the phase distribution function, $d\Gamma = dx dp$ is an infinitesimal element of phase space, k is the Boltzmann constant and $q \in \mathbf{R}$ is a real parameter, $q \in [0, \infty]$. The main results in this formalism for the microcanonical and canonical ensembles were obtained by us in detail in [17]. It was shown that if the parameter q is a universal constant, i.e. it takes identical values for different subsystems, then the Tsallis thermostatics violates the zeroth law of thermodynamics and the connection between statistical mechanics and equilibrium thermodynamics is lost. Now, following the method defined in [15, 16], we show that this violation can be removed. Note that we use general relations obtained in our previous work [17].

A. Microcanonical ensemble (E, V, z, N)

Let us consider an isolated thermodynamical system, specified by (E, V, z, N) . It is supposed that all dynamical systems in an equilibrium statistical ensemble have identical energy E within $\Delta E \ll E$. The phase distribution function and the statistical weight in the classical statistical mechanics can be written as [17]

$$f = \varrho^{\frac{z}{z+1}} = W^{-1} \Delta(H - E), \quad (3)$$

$$W = \int \Delta(H - E) d\Gamma, \quad (4)$$

where H is the Hamilton function and $\Delta(\varepsilon)$ is the function distinct from zero only in the interval $0 \leq \varepsilon \leq \Delta E$, where it is equal to unity. The thermodynamical variable of state z is expressed through the parameter q [15] as

$$z = \frac{q}{1 - q}. \quad (5)$$

Then, the entropy of the system in terms of the variables of state (E, V, z, N) takes the following form [17]

$$S = k(z + 1)[1 - e^{-S_G/kz}], \quad (6)$$

where $S_G = k \ln W$ is the usual Boltzmann-Gibbs entropy in the microcanonical ensemble.

In the microcanonical ensemble the division of a total system into two dynamically independent subsystems, $H = H_1 + H_2$, for finite number of particles, $N = N_1 + N_2$, leads to a convolution of the statistical weight W

$$W(E, V, N) = \frac{N_1! N_2!}{N!} \int_0^E dE_1 W_1(E_1, V, N_1) W_2(E - E_1, V, N_2), \quad (7)$$

where $W_i = \int \Delta(H_i - E_i) d\Gamma_i$. The statistical weight (4) does not factorize for finite energy E and finite number of particles N in a finite volume V . Hence, in the microcanonical ensemble even the Gibbs entropy is nonadditive, $S_G \neq S_{G,1} + S_{G,2}$, and consequently the Tsallis entropy is also nonadditive, $S \neq S_1 + S_2$. Only in the thermodynamic limit ($N \rightarrow \infty, E \rightarrow \infty, V \rightarrow \infty$ and $\varepsilon = E/N = \text{const}, v = V/N = \text{const}$) the statistical weight (7) factorizes, $W = W_1 W_2$. Factorizing requires the N -dependence $W = w^N$ with $w = w(\varepsilon, v)$ a function depending only on intensive variables of state. Then, the Gibbs entropy is additive, but for the Tsallis entropy there exist two possibilities: the parameter z can be either intensive or extensive.

First we present arguments in favor of treating the parameter z as an extensive variable. We inspect the equality of temperatures of two subsystems kept in equilibrium in the thermodynamical limit both by assuming a universal and an extensive value $z = N\bar{z}$.

The inverse temperature, by definition, is given as $1/T = \partial S/\partial E$. For the Tsallis definition of entropy it becomes

$$\frac{1}{T} = k \frac{z+1}{z} W^{-1/z} \frac{\partial \ln W}{\partial E}. \quad (8)$$

In the thermodynamical limit $W = W_1 W_2$ and $N = N_1 + N_2$ while specific values, like $\varepsilon = E/N$, are constant. The inverse temperature in this limit becomes

$$\frac{1}{T} = k \frac{z+1}{z} w^{-N/z} \frac{\partial \ln w}{\partial \varepsilon}. \quad (9)$$

Considering z as a universal constant, we arrive at an N -dependent temperature even in the thermodynamical limit, and therefore the equality of temperatures between large subsystems is spoiled: $T_1 \neq T_2 \neq T$. Moreover, the Tsallis entropy is nonadditive

$$S = S_1 + S_2 - \frac{1}{k(z+1)} S_1 S_2. \quad (10)$$

On the contrary if z is treated as an extensive variable itself, $z = N\tilde{z}$, then in the $N \rightarrow \infty$ limit equilibrium is established via

$$T_1 = T_2 = T, \quad \tilde{z}_1 = \tilde{z}_2 = \tilde{z} \quad (11)$$

and $S = S_1 + S_2$. One concludes that treating z as extensive the Tsallis entropy is additive in the thermodynamic limit and the zero law of thermodynamics is satisfied.

Investigating the thermodynamical functions of state in the thermodynamical limit amounts to expanding these functions in inverse powers of a large extensive parameter, for example, $1/N$ ($N \gg 1, |z| \gg 1, E \gg 1, V \gg 1$ and $\varepsilon = E/N = \text{const}, v = V/N = \text{const}, \tilde{z} = z/N = \text{const}$), while keeping ratios to N (specific values) finite [15]. In the thermodynamic limit the entropy (6) for large values of $|z| \gg 1$ can be written as

$$S = kz[1 - e^{-S_G/kz}]. \quad (12)$$

The Boltzmann-Gibbs entropy is an extensive function, $S_G(\lambda E, \lambda V, \lambda N) = \lambda S_G(E, V, N)$. Then, the Tsallis entropy in the microcanonical ensemble (12) is a first order homogeneous function of the variables E, V, z, N

$$S(\lambda E, \lambda V, \lambda z, \lambda N) = \lambda S(E, V, z, N). \quad (13)$$

A complete account of the derivation of thermodynamical relations based on Eq. (13) can be found in [15]. Here we augment only the main results. First, Euler's theorem for homogeneous functions can be applied to Eq. (13). We obtain

$$TS = E + pV + Xz - \mu N. \quad (14)$$

Then the partial derivatives are given as

$$\begin{aligned} \left(\frac{\partial S}{\partial E} \right)_{V,z,N} &= \frac{1}{T}, & \left(\frac{\partial S}{\partial V} \right)_{E,z,N} &= \frac{p}{T}, \\ \left(\frac{\partial S}{\partial z} \right)_{E,V,N} &= \frac{X}{T}, & \left(\frac{\partial S}{\partial N} \right)_{E,V,z} &= -\frac{\mu}{T}. \end{aligned} \quad (15)$$

and finally Euler's theorem provides the fundamental differential form,

$$TdS = dE + pdV + Xdz - \mu dN, \quad (16)$$

as well as the Gibbs-Duhem relation

$$SdT = Vdp + zdX - Nd\mu, \quad (17)$$

and consequently the first and the second laws of macroscopic thermodynamics. The temperature T , the variable X , the pressure and the chemical potential of the system are expressed through the variables of the Gibbs statistics, as follows:

$$T = T_G e^{S_G/kz}, \quad (18)$$

$$X = kT_G [e^{S_G/kz} - (1 + S_G/kz)] \quad (19)$$

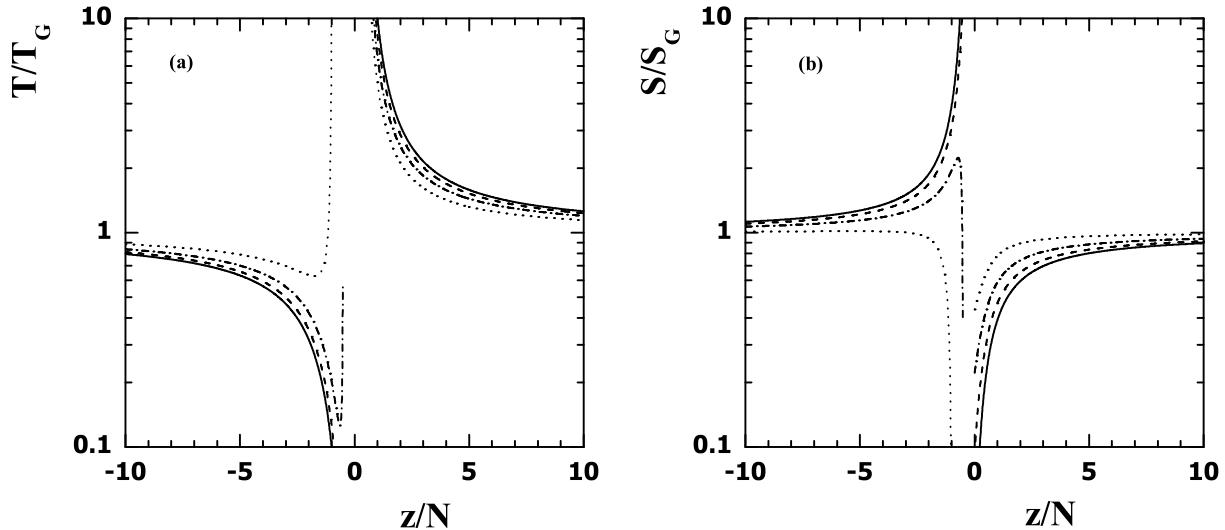


FIG. 1: The dependence of the ratio of temperature T to the Gibbs temperature T_G (a) and the ratio of the entropy S to the Gibbs entropy S_G (b) on the specific $\tilde{z} = z/N$ for the classical non-relativistic ideal gas of nucleons in the microcanonical ensemble at the values of the specific energy $\varepsilon = 50$ MeV and the specific volume $v = 3/\rho_0$. The solid lines represent the calculations treating z as extensive. The dotted, dash-dotted and dashed lines correspond to the calculations considering z as a universal constant for the number of particles $N = 1, 2$ and 5 , respectively.

and $p = p_G$, $\mu = \mu_G$, respectively. Thus, the temperature T , the pressure p , the chemical potential μ , and the quantity X are homogeneous functions of the variables E, V, z, N of order zero: They are intensive variables. Note that in the thermodynamical limit all relations in the microcanonical ensemble concerning Wang's formalism of Tsallis thermostatics completely coincide with the ones of the original Tsallis thermostatics given in [15]. Moreover, all conclusions relative to the Tsallis statistics in the microcanonical ensemble given in [15] are also preserved in the case of the Wang's formalism in the thermodynamical limit. Therefore, whenever the parameter $z = q/(1-q)$ is an extensive variable of state, then in the thermodynamical limit the principle of additivity and the zeroth law of thermodynamics are valid. In particular the Tsallis entropy is extensive and the temperature is intensive. In this case the Tsallis statistics completely satisfies all postulates of equilibrium thermodynamics. An evaluation of thermodynamical identities in the microcanonical ensemble is provided by the Euler theorem.

In the thermodynamical limit in the microcanonical ensemble results for an ideal gas of identical particles are coincident in both Wang's and Tsallis' original formalisms. Expressions for the ideal gas in the microcanonical ensemble for the Tsallis statistics can be found in [15]. Here we present relations for the temperature and entropy either considering the parameter z as a universal constant or as a extensive variable of state. The one-particle statistical weight for the classical non-relativistic ideal gas is given by

$$w = v \left(\frac{m\varepsilon e^{5/3}}{3\pi\hbar^2} \right)^{3/2}. \quad (20)$$

For a universal $z = z_1 = z_2$ in the thermodynamical limit the temperature (9) and entropy (6) are given by

$$T = \frac{2\varepsilon}{3k} \frac{z}{z+1} w^{N/z}, \quad (21)$$

$$S = k(z+1) \left[1 - w^{-N/z} \right]. \quad (22)$$

This example illustrates that in the original Tsallis' approach the temperature is not-intensive and entropy is not-extensive. On the other hand if the parameter z is extensive, $z = N\tilde{z}$, then in the thermodynamical limit ($N \gg 1$) the temperature (9) and entropy (6) take the form

$$T = \frac{2\varepsilon}{3k} w^{1/\tilde{z}}, \quad (23)$$

$$S = k\tilde{z}N \left[1 - w^{-1/\tilde{z}} \right]. \quad (24)$$

In this case the temperature is intensive and the entropy is extensive, in complete agreement with requirements of equilibrium thermodynamics.

B. Canonical ensemble (T, V, z, N)

In the canonical ensemble in Wang's formalism of Tsallis thermostatics the equilibrium phase distribution function can be written as [17]

$$\varrho^{\frac{z}{z+1}} = \left[1 + \frac{z}{(z+1)^2} \frac{\Lambda - H}{kT} \right]^z, \quad (25)$$

where $\Lambda = \langle H \rangle - \frac{z+1}{z} TS$ is determined from the normalization condition (1)

$$\int \left[1 + \frac{z}{(z+1)^2} \frac{\Lambda - H}{kT} \right]^z d\Gamma = 1. \quad (26)$$

Thus Λ , by solving Eq. (26), is a function of the variables of state, $\Lambda = \Lambda(T, V, z, N)$. The expectation value $\langle A \rangle$ of a dynamical variable A can be computed as (cf. (2)):

$$\langle A \rangle = \int A \left[1 + \frac{z}{(z+1)^2} \frac{\Lambda - H}{kT} \right]^z d\Gamma. \quad (27)$$

The entropy takes the form

$$S = \frac{z}{z+1} \frac{\langle H \rangle - \Lambda}{T}, \quad (28)$$

and the free energy is unambiguously determined by the functions Λ and $\langle H \rangle$ as the Legendre transform of energy with respect to the entropy of the system

$$F \equiv \langle H \rangle - TS = \frac{\langle H \rangle + z\Lambda}{z+1}. \quad (29)$$

In the canonical ensemble of Gibbs statistics the division of a total system into two dynamically independent subsystems, $H = H_1 + H_2$, for finite number of particles, $N = N_1 + N_2$, does not lead to factorization of the distribution function, $\varrho_G \neq \varrho_{G,1}\varrho_{G,2}$, due to the relation

$$Z(T, V, N) = \frac{N_1!N_2!}{N!} Z_1(T, V, N_1)Z_2(T, V, N_2), \quad (30)$$

where $Z = \int d\Gamma \exp(-H/T)$ and $\varrho_G = Z^{-1} \exp(-H/T)$ are the canonical partition function and the distribution function, respectively. The partition function (30) does not factorize because of the Gibbs factor $N!$ and finite volume V . In the canonical ensemble even the Gibbs entropy is not-additive for finite systems, $S_G \neq S_{G,1} + S_{G,2}$. Hence for dynamically independent subsystems at finite values of N, V we have statistical dependence. In case of Tsallis statistics we have equivalent results for the distribution function, $\varrho \neq \varrho_1\varrho_2$, and entropy $S \neq S_1 + S_2$ at finite values of N, V . In the thermodynamical limit ($N \rightarrow \infty, V \rightarrow \infty, v = \text{const}$) the Gibbs and Tsallis approaches differ essentially. The Gibbs partition function factorizes, $Z_G = Z_{G,1}Z_{G,2}$, it leads to $Z_G = \tilde{Z}_G^N$, where $\tilde{Z}_G = \tilde{Z}_G(T, v)$ is an intensive function of intensive variables of state only. For the Tsallis statistics there are two possibilities. First, if the parameter z is a universal constant then even in the thermodynamic limit ($N \rightarrow \infty, V \rightarrow \infty, v = \text{const}, z = \text{const}$) the distribution function (25) does not factorize

$$\varrho^{\frac{z}{z+1}} \neq \varrho_1^{\frac{z}{z+1}} \varrho_2^{\frac{z}{z+1}}. \quad (31)$$

The entropy is not-additive and Tsallis formula (10) cannot be applied. On the other hand, keeping $\tilde{z} = z/N$ constant in the thermodynamic limit, two dynamically independent subsystems are independent also statistically. In this case the distribution function (25) can be written as

$$\varrho = \left[1 + \frac{1}{\tilde{z}} \frac{\lambda - \tilde{H}}{T} \right]^{\tilde{z}N}, \quad (32)$$

where $\lambda = \Lambda/N$ is an intensive variable, $\lambda_1 = \lambda_2 = \lambda$. If the reduced Hamiltonian (energy per particle), $\tilde{H} = H/N$, is assumed to be an intensive function, $\tilde{H}_1 \sim \tilde{H}_2 \sim \tilde{H}$, then for equal temperatures, $T_1 = T_2 = T$, the distribution function (32) in fact factorizes and the entropy is additive

$$\varrho = \varrho_1\varrho_2, \quad \tilde{z} = \tilde{z}_1 = \tilde{z}_2, \quad (33)$$

$$S = S_1 + S_2. \quad (34)$$

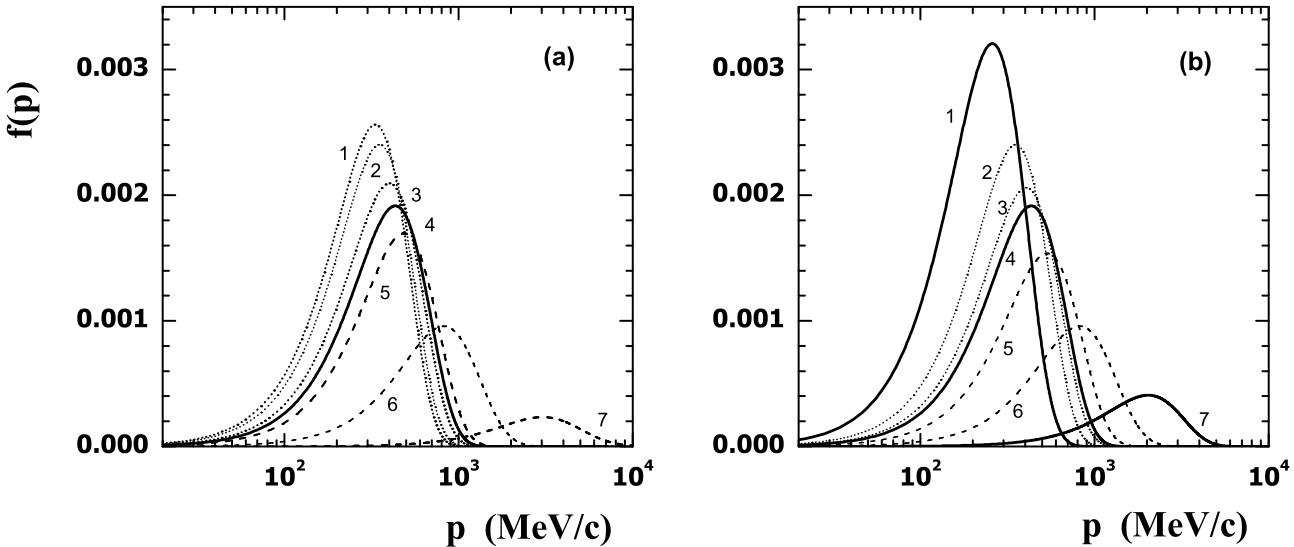


FIG. 2: The single-particle distribution function for classical non-relativistic ideal gas of nucleons in Wang's formalism of Tsallis statistics at the temperature $T = 100$ MeV and the specific volume $v = 3/\rho_0$ for two treatments of a parameter z : (a) a universal constant [17] for $N = 10$ and different values of $z = 20, 30, 100, \pm\infty, -100, -30$ and -20 (the curves 1, 2, 3, 4, 5, 6 and 7, respectively), (b) an extensive variable of state [16] in thermodynamic limit for different values of the specific $\tilde{z} = 3, \pm\infty, -3$ (solid lines 1, 4, 7, respectively). The dotted and dashed lines (b) correspond to the calculations considering z as an universal constant [17] for $\tilde{z} = 3$ and number of particles $N = 10, 50$ (lines 2,3) and $\tilde{z} = -3$ and $N = 50, 10$ (lines 5,6). The line 4 on both panels corresponds to the conventional Boltzmann-Gibbs statistics.

Note that the distribution function (32) is expressed in the form, $\varrho = \tilde{\varrho}^N$.

Let us now deduce the fundamental equation of thermodynamics for Tsallis statistics with an extensive z . Applying the total differential operator with respect to all ensemble variables (T, V, z, N) on the entropy S , the energy $\langle H \rangle$ and the norm equation (1), and using Eq. (25) and the parametric dependence of the Hamilton function H on the variables V , N and z one finds the fundamental equation of thermodynamics [5, 16, 17]

$$TdS = d\langle H \rangle + pdV + Xdz - \mu dN, \quad (35)$$

where

$$p = \int d\Gamma \varrho^{\frac{z}{z+1}} \left(-\frac{\partial H}{\partial V} \right)_{T,z,N} = \left\langle -\frac{\partial H}{\partial V} \right\rangle, \quad (36)$$

$$\mu = \int d\Gamma \varrho^{\frac{z}{z+1}} \left(\frac{\partial H}{\partial N} \right)_{T,V,z} = \left\langle \frac{\partial H}{\partial N} \right\rangle, \quad (37)$$

$$X = \int d\Gamma \varrho^{\frac{z}{z+1}} \left\{ kT \left[1 - \varrho^{\frac{1}{z+1}} \left(1 - \frac{z+1}{z} \ln \varrho^{\frac{1}{z+1}} \right) \right] + \left(-\frac{\partial H}{\partial z} \right)_{T,V,N} \right\}. \quad (38)$$

Here, the property of the Hamilton function $(\partial H/\partial T)_{V,z,N} = 0$ was assumed. Using Eqs. (29) and (35) we get a differential formula for the free energy

$$dF = -SdT - pdV - Xdz + \mu dN. \quad (39)$$

In the canonical ensemble the thermodynamical limit means to let $N \rightarrow \infty$, while $v = V/N$ and $\tilde{z} = z/N$ kept constant. We make an expansion of functions of state in powers of the small parameter $1/N$ (with $N \gg 1$ it is $|z| \gg 1$ and $V \gg 1$) [16]. Note that in the thermodynamical limit all relations of the canonical ensemble derived here completely coincide with ones of the original Tsallis thermostatics given in [16]. Moreover, all results for the ideal gas in the canonical ensemble for both formalisms are equivalent. To supply evidence, we should rewrite the expressions for the perfect gas in the canonical ensemble in Wang's formalism given in [17] in terms of the variable z (5), and then impose the thermodynamical limit. Expressions for the ideal gas in the canonical ensemble for the Tsallis statistics can be found in [16]. Thus all conclusions relative to Tsallis statistics in the canonical ensemble given in [16] are preserved in the case of Wang's formalism in the thermodynamical limit. The statistical mechanics based on Tsallis entropy completely satisfies all requirements of equilibrium thermodynamics in the canonical ensemble in the thermodynamical limit. It was shown that all functions of state of the system are the homogeneous functions of the first degree, extensive, or the homogeneous functions of the zero degree, intensive. In particular, the temperature is an intensive variable and thus provides implementation of the zero law of thermodynamics.

III. RÉNYI THERMOSTATICS

The Rényi thermostatics is based on Rényi's definition of statistical entropy with a usual norm equation for the phase distribution function

$$S = k \frac{\ln (\int \varrho^q d\Gamma)}{1 - q}, \quad \int \varrho d\Gamma = 1. \quad (40)$$

The expectation value of a dynamical variable A is given as

$$\langle A \rangle = \int A \varrho d\Gamma. \quad (41)$$

Note that the main relations for Rényi thermostatics were derived in [17]. Here our investigation proceeds with the description of the entropic parameter q through a thermodynamical variable of state z ,

$$z = \frac{1}{q - 1}. \quad (42)$$

This allows us to introduce a correct thermodynamical limit consistent with the macroscopic laws of equilibrium thermodynamics. We rewrite all expressions of the Rényi thermostatics given in [17] in terms of the variable z and implement the thermodynamical limit described above.

A. Microcanonical ensemble (E, V, z, N)

In the microcanonical ensemble of the Rényi thermostatics the phase distribution function and the statistical weight are [17]

$$\varrho = W^{-1} \Delta(H - E), \quad (43)$$

$$W = \int \Delta(H - E) d\Gamma. \quad (44)$$

Then, the Rényi entropy (40) is reduced to the familiar expression

$$S = k \ln W \equiv S_G, \quad (45)$$

where S_G is the Gibbs entropy which does not depend on the variable of the state z . Hence, all expressions of the Rényi statistics in the microcanonical ensemble are equivalent with ones of the conventional Gibbs statistics. The conclusions concerning the microcanonical ensemble of the corresponding Gibbs thermostatics can be generalized to apply to the Rényi one. The Rényi entropy in the microcanonical ensemble (45) is a first order homogeneous function of variables E, V, N

$$S(\lambda E, \lambda V, \lambda N) = \lambda S(E, V, N). \quad (46)$$

Eq. (46) provides the Euler theorem, the fundamental equation of thermodynamics, the Gibbs-Duhem relation and consequently the first and the second laws of the macroscopic thermodynamics which all are independent of the parameter q . All functions of state in the thermodynamical limit are either intensive or extensive according to requirements of equilibrium thermodynamics. Thus, the principle of additivity and the zero law of thermodynamics are valid. The Rényi statistics in the microcanonical ensemble completely satisfies all requirements of the equilibrium thermodynamics. Note that specific expressions for the ideal gas in the microcanonical ensemble for the Gibbs statistics can be found in [15].

B. Canonical ensemble (T, V, z, N)

In the canonical ensemble of the Rényi thermostatics the phase distribution function depends on two, directly unknown, variables S and $\langle H \rangle$ [17]

$$\varrho = e^{-S/k} \left[1 + \frac{1}{z+1} \frac{\langle H \rangle - H}{kT} \right]^z. \quad (47)$$

In order to normalize the phase distribution function (47) two equations have to be applied [17]: $\int d\Gamma \varrho = 1$ and $\langle H \rangle = \int d\Gamma H \varrho$. We obtain the entropy and the energy of the system as functions of the variables of state, $S = S(T, V, z, N)$ and $\langle H \rangle = \langle H \rangle(T, V, z, N)$, respectively. The free energy can be written as

$$F = \langle H \rangle - TS. \quad (48)$$

The expectation value of a dynamical variable A in this case is given by

$$\langle A \rangle = e^{-S/k} \int A \left[1 + \frac{1}{z+1} \frac{\langle H \rangle - H}{kT} \right]^z d\Gamma. \quad (49)$$

Let us derive the fundamental equation of thermodynamics at fixed values of the variables of state (T, V, z, N) for Rényi statistics with an extensive z . Following the procedure given above by virtue of the parametric dependence of the Hamilton function H on the variables V , z and N only we obtain the fundamental equation of thermodynamics [5, 16, 17]

$$TdS = d\langle H \rangle + pdV + Xdz - \mu dN, \quad (50)$$

with

$$p = \int d\Gamma \varrho \left(-\frac{\partial H}{\partial V} \right)_{T,z,N} = \left\langle -\frac{\partial H}{\partial V} \right\rangle, \quad (51)$$

$$\mu = \int d\Gamma \varrho \left(\frac{\partial H}{\partial N} \right)_{T,V,z} = \left\langle \frac{\partial H}{\partial N} \right\rangle, \quad (52)$$

$$X = kT \left[\frac{S}{kz} + e^{\frac{S}{kz}} \int d\Gamma \varrho^{\frac{z+1}{z}} \ln \varrho^{\frac{1}{z}} \right] + \int d\Gamma \varrho \left(-\frac{\partial H}{\partial z} \right)_{T,V,N}. \quad (53)$$

Here the property of the Hamilton function, $(\partial H / \partial T)_{V,z,N} = 0$, was used. For the Rényi thermostatics Eq. (39) is valid and the functions of state X, p and μ can be calculated from the corresponding thermodynamical relations. It is important to note that in the thermodynamical limit all expressions of the ideal gas [17] are exactly equivalent with ones of the Gibbs statistics both the thermodynamical functions of state and the one-particle distribution function. In the last case in the thermodynamical limit the non-relativistic single-particle distribution function is reduced to the Maxwell-Boltzmann distribution $f(\vec{p}) = (2\pi mkT)^{-3/2} \exp(-\vec{p}^2/2mkT)$ upon rewriting the expressions for the ideal gas in the canonical ensemble for the Rényi statistics in terms of the variable z (42), as it is given in [17], and impose then the thermodynamical limit condition. Specific expressions for the ideal gas in the canonical ensemble for the Gibbs statistics can be found in [16]. Thus, all conclusions concerning the canonical ensemble for the Gibbs statistics on the particular example of the ideal gas can be generalized to apply to the Rényi thermostatics. The Rényi statistics in the canonical ensemble completely satisfies all requirements of equilibrium thermodynamics. In particular, the principle of additivity, the zeroth, the first, the second and the third laws of thermodynamics are valid.

IV. CONCLUSION

In this paper Wang's formalism of the Tsallis statistics was compared to the Rényi one in the thermodynamical limit. An unambiguous connection between both statistical mechanics and the equilibrium thermodynamics are revealed. For this purpose expressions obtained in Wang's formalism of Tsallis statistics and the Rényi one were rewritten in terms of a new extensive variable of state, z , related to the entropic parameter q . The functions of state were regularized by applying the limiting procedure of the thermodynamical limit. We obtained that in this limit Wang's formalism of the Tsallis statistics in the terms of the extensive variable of state z completely coincide with the original Tsallis thermostatics in both the microcanonical and the canonical ensembles. However, the Rényi statistics resembles the usual Boltzmann-Gibbs thermodynamics. In the microcanonical ensemble we proved for both the Rényi and Wang's statistics that in the thermodynamical limit all laws of thermodynamics, in particular the zeroth law, the principle of additivity, the Euler theorem, the fundamental equation of thermodynamics and the Gibbs-Duhem relation are valid. To put it simply both the Tsallis and the Rényi entropies are extensive in the thermodynamical limit, provided one composes subsystems with the proper, and hence different, q -values.

In the canonical ensemble, however, only the fundamental equation of thermodynamics, the first and the second laws were derived in general terms. For demonstrating further principles of equilibrium thermodynamics in the framework of the canonical ensemble, both for the Rényi statistics and for Wang's formalism of Tsallis statistics, exact analytical results were utilized for the ideal gas of identical particles. It was shown for this

particular example that in the thermodynamical limit for both thermostatics the main thermodynamical equations, the zeroth law and the principle of additivity are satisfied. All functions of state are either extensive or intensive. Moreover, in the canonical ensemble in the thermodynamical limit both the Tsallis and the Rényi entropies are extensive. For the ideal gas of identical particles in both thermostatics the equivalence of the canonical and the microcanonical ensembles in the thermodynamical limit was demonstrated. This is a very important property to be verified for the self-consistent definition of any statistical mechanics.

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